1) $y = \sqrt{\left(\frac{1+t^2}{1-t^2}\right)^2 - 1}$

$$\frac{dy}{dt} = \frac{d}{dt}\left\{\left(\frac{1+t^2}{1-t^2}\right)^2 - 1\right\}^{1/2}$$

$$= \frac{d}{dt}\left\{\frac{(1+t^2)^2 - (1-t^2)^2}{(1-t^2)^2}\right\}^{1/2}$$

$$= \frac{d}{dt}\left\{\frac{4t^2}{(1-t^2)^2}\right\}^{1/2}$$

$(\because (a+b)^2 - (a-b)^2 = 4ab)$

$$= \frac{d}{dt}\frac{2t}{1-t^2}$$

$$= 2\frac{d}{dt}\frac{t}{1-t^2}$$

$$= 2\,\frac{1(1-t^2) + 2t \cdot t}{(1-t^2)^2}$$

$$= 2\frac{(1+t^2)}{(1-t^2)^2}$$

2) $y = x^2 \cos^{-1}\frac{\sqrt{x}-1}{\sqrt{x}+1} + x^2 \operatorname{cosec}^{-1}\frac{\sqrt{x}+1}{\sqrt{x}-1}$

$$\boxed{\begin{aligned} \operatorname{cosec}^{-1}x &= \sin^{-1}1/x \\ \sin^{-1}x + \cos^{-1}x &= \pi/2 \end{aligned}}$$

$$y = x^2 \cos^{-1}\frac{\sqrt{x}-1}{\sqrt{x}+1} + x^2 \sin^{-1}\frac{\sqrt{x}-1}{\sqrt{x}+1}$$

$$= x^2\left(\cos^{-1}\frac{\sqrt{x}-1}{\sqrt{x}+1} + \sin^{-1}\frac{\sqrt{x}-1}{\sqrt{x}+1}\right)$$

$$= x^2 \cdot \pi/2$$

$$dy/dx = \frac{d}{dx}\pi x^2/2 = \pi/2 \frac{d}{dx}x^2$$

$$= \pi/\cancel{2}\,\cancel{2}x = \pi x$$

3) if $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$

then find dy/dx

Ans $\sqrt{1-x^2} + \sqrt{1-y^2} = a(x-y)$

Put $x = \sin\alpha$, $y = \sin\beta$

$\sqrt{1-\sin^2\alpha} + \sqrt{1-\sin^2\beta} = a(\sin\alpha - \sin\beta)$

$\Rightarrow \cos\alpha + \cos\beta = a(\sin\alpha - \sin\beta)$

$\Rightarrow 2\cos\cancel{\frac{\alpha+\beta}{2}}\cos\frac{\alpha-\beta}{2} = 2a\cos\cancel{\frac{\alpha+\beta}{2}}\sin\frac{\alpha-\beta}{2}$

$\Rightarrow \cos\frac{(\alpha-\beta)}{2} = a\sin\left(\frac{\alpha-\beta}{2}\right)$

$$\Rightarrow \frac{\cos\left(\frac{\alpha-\beta}{2}\right)}{\sin\left(\frac{\alpha-\beta}{2}\right)} = a$$

$$\Rightarrow \cot\left(\frac{\alpha-\beta}{2}\right) = a$$

$$\Rightarrow \frac{\alpha-\beta}{2} = \cot^{-1} a$$

$$\Rightarrow \alpha-\beta = 2\cot^{-1} a$$

$$\Rightarrow \sin^{-1} x - \sin^{-1} y = 2\cot^{-1} a$$

$$\frac{d}{dx}\left(\sin^{-1} x - \sin^{-1} y\right) = \frac{d}{dx}\left(2\cot^{-1} a\right)$$

$$\Rightarrow \frac{1}{\sqrt{1-x^2}} - \frac{1}{\sqrt{1-y^2}}\frac{dy}{dx} = 0$$

$$\Rightarrow \frac{1}{\sqrt{1-x^2}} = \frac{1}{\sqrt{1-y^2}}\frac{dy}{dx}$$

$$\Rightarrow \frac{dy}{dx} = \sqrt{\frac{1-y^2}{1-x^2}}$$

4. if $x\sqrt{1+y} + y\sqrt{1+x} = 0$

then find dy/dx

Ans. $x\sqrt{1+y} + y\sqrt{1+x} = 0$

$\Rightarrow x\sqrt{1+y} = -y\sqrt{1+x}$

Squaring both sides

$x^2(1+y) = y^2(1+x)$

$\Rightarrow x^2 + x^2y = y^2 + y^2x$

$\Rightarrow x^2 - y^2 = y^2x - x^2y = xy(y-x)$

$\Rightarrow (x+y)(x-y) = xy(y-x)$

$\Rightarrow x+y = -xy$

$\Rightarrow -x = y + xy$

$\Rightarrow -x = y(1+x)$

$\Rightarrow y = -\frac{x}{1+x}$

$\Rightarrow \frac{dy}{dx} = \frac{d}{dx}\left(-\frac{x}{1+x}\right)$

$= \frac{(-1)(1+x)+x}{(1+x)^2}$

$= \frac{-1-x+x}{(1+x)^2} = \frac{-1}{(1+x)^2}$.

5. Differentiate $\operatorname{cosec}(\cot^{-1}x)$ w.r.t. $\sec(\tan^{-1}x)$

$y = \operatorname{cosec}(\cot^{-1}x) \quad z = \sec(\tan^{-1}x)$

Let $\cot^{-1}x = p$

$\Rightarrow x = \cot p$

$\therefore y = \operatorname{cosec} p$

$dy/dp = \frac{d}{dp}(\operatorname{cosec} p) = -\operatorname{cosec} p \cot p$

$$p = \cot^{-1} x$$

$$\Rightarrow \frac{dp}{dx} = \frac{d}{dx} \cot^{-1} x = -\frac{1}{1+x^2}$$

$$\cancel{\frac{dy}{dp} = \left(\frac{dy}{dx}\right) \Big/ \left(\frac{dp}{dx}\right)}$$

$$\frac{dy}{dp} \cdot \frac{dp}{dx} = \frac{dy}{dx}$$

$$\therefore \frac{dy}{dx} = -\operatorname{cosec} p \cot p \left(-\frac{1}{1+x^2}\right)$$

$$= \frac{\sqrt{1+\cot^2 p}\ \cot p}{1+x^2}$$

$$= \frac{\sqrt{1+x^2}\ \cot p}{1+x^2} \quad (\because \cot p = x)$$

$$= \frac{\cot p}{\sqrt{1+x^2}} = \frac{x}{\sqrt{1+x^2}}$$

Put $\tan^{-1}x = q$

$\Rightarrow x = \tan q$

$\therefore z = \sec q$

$\frac{dz}{dq} = \frac{d}{dq}\sec q = \sec q \cdot \tan q$

$x = \tan q$

$q = \tan^{-1}x$

$\frac{dq}{dx} = \frac{d}{dx}\tan^{-1}x = \frac{1}{1+x^2}$

$\frac{dz}{dx} = \frac{dz}{dq} \cdot \frac{dq}{dx} = \sec q \cdot \tan q \cdot \frac{1}{1+x^2}$

$= \frac{\sqrt{1+\tan^2 q}\ \tan q}{1+x^2}$

$= \frac{\sqrt{1+x^2}\ x}{1+x^2}$ [$\because x = \tan q$]

$= \frac{x}{\sqrt{1+x^2}}$

$$\frac{dy}{dx} = \frac{dy/dx}{\frac{dz}{dx}} = \frac{\frac{x}{\sqrt{1+x^2}}}{\frac{x}{\sqrt{1+x^2}}} = 1$$

6. $y = \sin^{-1}\left(2ax\sqrt{1-a^2x^2}\right)$

Find dy/dx

Ans.

$$y = \sin^{-1}\left(2ax\sqrt{1-a^2x^2}\right)$$

$$= \sin^{-1}\sqrt{4a^2x^2(1-a^2x^2)}$$

Put $ax = \sin\theta$

$$\therefore y = \sin^{-1}\sqrt{4\sin^2\theta(1-\sin^2\theta)}$$

$$= \sin^{-1}\sqrt{4\sin^2\theta\cos^2\theta}$$

$$= \sin^{-1}\sqrt{(2\sin\theta\cos\theta)^2}$$

$$= \sin^{-1} 2\sin\theta\cos\theta$$

$$= \sin^{-1}\sin 2\theta$$

$$= 2\theta$$

$\therefore ax = \sin\theta$

$\Rightarrow \sin^{-1} ax = \theta$

$\therefore y = 2\sin^{-1} ax$

$$dy/dx = \frac{d}{dx}(2\sin^{-1} ax)$$

$$= 2\frac{d}{dx}\sin^{-1} ax$$

$$= 2\frac{d\sin^{-1} ax}{d(ax)} \times \frac{d(ax)}{dx}$$

$$= \frac{2a}{\sqrt{1-a^2x^2}}$$