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DEPARTMENT OF MATH AND SCIENCE

MATHEMATICS

Lecture Notes 2026-27

Classroom Lecture Notes on

Unit-3 Complex Numbers

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GOVERNMENT POLYTECHNIC, DEBAGADA

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1 Introduction

Complex numbers extend the real number system. They are particularly useful when an algebraic equation does not have a real solution.

A complex number is written in the form

$$z = x + iy,$$

where

$$x, y \in \mathbb{R}$$

and

$$i = \sqrt{-1}.$$

The unit contains two closely related parts. The first part deals with complex numbers and their different representations. The second part deals with polynomial fractions and partial fraction decomposition.

The main topics are:

- Basic concept of complex numbers
- Real and imaginary parts
- Algebraic operations
- Properties of complex numbers
- Equality of two complex numbers
- Conjugate
- Modulus
- Argument or amplitude
- Cartesian representation
- Polar representation
- Conversion between forms
- De Moivre's theorem
- Applications of De Moivre's theorem

- Polynomial fractions
- Proper and improper fractions
- Partial fractions

Important Point

The standard algebraic form of a complex number is

$$z = x + iy.$$

2 Basic Concept of Complex Numbers

2.1 Definition

Definition

A complex number is a number of the form

$$z = x + iy,$$

where $x, y \in \mathbb{R}$ and

$$i = \sqrt{-1}.$$

The set of all complex numbers is denoted by $\mathbb{C}$.

Some examples are

$$5 + 3i, \quad 1 - i, \quad 4i, \quad 7.$$

Every real number can be regarded as a complex number.

For example,

$$7 = 7 + 0i.$$

Similarly,

$$4i = 0 + 4i.$$

Thus real numbers and purely imaginary numbers are special cases of complex numbers.

2.2 Powers of i

The fundamental relation is

$$i^2 = -1.$$

Multiplying successively by i ,

$$i^3 = i^2 i = -i,$$

and

$$i^4 = i^2 i^2 = (-1)(-1) = 1.$$

Therefore the powers repeat in a cycle:

$$i, \quad -1, \quad -i, \quad 1.$$

Hence,

$$i^{4n} = 1,$$

$$i^{4n+1} = i,$$

$$i^{4n+2} = -1,$$

$$i^{4n+3} = -i.$$

Solved Example

Find i^{27} .

We write

$$27 = 4(6) + 3.$$

Therefore,

$$i^{27} = i^{4(6)+3} = -i.$$

Hence,

$$\boxed{i^{27} = -i}.$$

Important Point

For a large power of i , divide the exponent by 4 and use the remainder.

3 Real and Imaginary Parts

Let

$$z = x + iy.$$

The real part is denoted by

$$\Re(z) = x$$

and the imaginary part is denoted by

$$\Im(z) = y.$$

Definition

If

$$z = x + iy,$$

then x is the real part and y is the imaginary part of z .

3.1 Purely Real Number

If

$$\Im(z) = 0,$$

then

$$z = x$$

is a purely real number.

Example:

$$z = 5 + 0i = 5.$$

3.2 Purely Imaginary Number

If

$$\Re(z) = 0,$$

then

$$z = iy$$

is purely imaginary.

Example:

$$z = 0 + 6i = 6i.$$

Solved Example

For

$$z = 8 - 5i,$$

we have

$$\Re(z) = 8$$

and

$$\Im(z) = -5.$$

4 Algebraic Operations on Complex Numbers

Let

$$z_1 = a + ib$$

and

$$z_2 = c + id.$$

4.1 Addition

We have

$$z_1 + z_2 = (a + c) + i(b + d).$$

Solved Example

Find

$$(3 + 4i) + (5 - 2i).$$

Therefore,

$$(3 + 4i) + (5 - 2i) = (3 + 5) + (4 - 2)i$$

and hence

$$8 + 2i.$$

4.2 Subtraction

$$z_1 - z_2 = (a - c) + i(b - d).$$

For example,

$$(7 + 5i) - (2 + 3i) = 5 + 2i.$$

4.3 Multiplication

$$(a + ib)(c + id) = ac + iad + ibc + i^2bd.$$

Since

$$i^2 = -1,$$

we get

$$(a + ib)(c + id) = (ac - bd) + i(ad + bc).$$

Solved Example

Multiply

$$(2 + 3i)(4 - i).$$

Expanding,

$$= 8 - 2i + 12i - 3i^2.$$

Since

$$i^2 = -1,$$

we get

$$= 8 + 10i + 3.$$

Therefore,

$$11 + 10i.$$

4.4 Division

Consider

$$\frac{a + ib}{c + id}.$$

Multiply numerator and denominator by the conjugate

$$c - id.$$

Thus,

$$\frac{a + ib}{c + id} = \frac{(a + ib)(c - id)}{(c + id)(c - id)}.$$

The denominator becomes

$$c^2 + d^2.$$

Hence,

$$\frac{a + ib}{c + id} = \frac{ac + bd}{c^2 + d^2} + i \frac{bc - ad}{c^2 + d^2}.$$

Solved Example

Simplify

$$\frac{3 + 2i}{1 - i}.$$

Multiply by the conjugate $1 + i$:

$$\frac{3 + 2i}{1 - i} = \frac{(3 + 2i)(1 + i)}{(1 - i)(1 + i)}.$$

The numerator is

$$3 + 3i + 2i + 2i^2 = 1 + 5i.$$

The denominator is

$$1 - i^2 = 2.$$

Therefore,

$$\frac{3 + 2i}{1 - i} = \frac{1}{2} + \frac{5}{2}i.$$

5 Properties of Complex Numbers

For complex numbers z_1, z_2, z_3 ,

$$z_1 + z_2 = z_2 + z_1$$

and

$$z_1 z_2 = z_2 z_1.$$

Thus addition and multiplication are commutative.

Also,

$$(z_1 + z_2) + z_3 = z_1 + (z_2 + z_3),$$

and

$$(z_1 z_2) z_3 = z_1 (z_2 z_3).$$

Multiplication is distributive:

$$z_1(z_2 + z_3) = z_1 z_2 + z_1 z_3.$$

These properties allow complex numbers to be manipulated using ordinary algebraic rules together with

$$i^2 = -1.$$

6 Equality of Two Complex Numbers

Let

$$z_1 = x_1 + iy_1$$

and

$$z_2 = x_2 + iy_2.$$

Then

$$z_1 = z_2$$

if and only if

$$x_1 = x_2$$

and

$$y_1 = y_2.$$

Solved Example

Solve

$$(x + iy)(2 - 3i) = 4 + i.$$

Expanding,

$$2x - 3xi + 2yi - 3yi^2 = 4 + i.$$

Since $i^2 = -1$,

$$(2x + 3y) + (2y - 3x)i = 4 + i.$$

Comparing real and imaginary parts,

$$2x + 3y = 4,$$

$$2y - 3x = 1.$$

Solving these equations gives

$$\boxed{x = \frac{5}{13}}, \quad \boxed{y = \frac{14}{13}}.$$

Important Point

Two complex numbers are equal only when both their real parts and imaginary parts are equal.

7 Conjugate of a Complex Number

Definition

If

$$z = a + ib,$$

then the conjugate of z is

$$\bar{z} = a - ib.$$

Examples:

$$\overline{(3 + 4i)} = 3 - 4i,$$

$$\overline{(5 - 2i)} = 5 + 2i.$$

Important identities are

$$z + \bar{z} = 2\Re(z),$$

$$z - \bar{z} = 2i\Im(z),$$

and

$$z\bar{z} = |z|^2.$$

The conjugate is particularly useful in the division of complex numbers.

8 Modulus of a Complex Number

Let

$$z = a + ib.$$

Definition

The modulus of z is

$$|z| = \sqrt{a^2 + b^2}.$$

Geometrically, $|z|$ is the distance of the point representing z from the origin.

Since

$$z\bar{z} = (a + ib)(a - ib),$$

we obtain

$$z\bar{z} = a^2 + b^2.$$

Therefore,

$$\boxed{z\bar{z} = |z|^2}.$$

8.1 Properties of Modulus

$$|z| \geq 0,$$

$$|z| = 0 \iff z = 0,$$

$$|\bar{z}| = |z|,$$

$$|z_1 z_2| = |z_1| |z_2|,$$

and

$$\left| \frac{z_1}{z_2} \right| = \frac{|z_1|}{|z_2|}, \quad z_2 \neq 0.$$

Solved Example

Find the modulus of

$$z = 6 - 8i.$$

We have

$$|z| = \sqrt{6^2 + (-8)^2} = \sqrt{36 + 64} = 10.$$

Therefore,

$$\boxed{|z| = 10}.$$

Important Point

A complex number satisfying

$$|z| = 1$$

is called a unimodular complex number.

9 Argument or Amplitude

Let

$$z = a + ib.$$

If the point representing z makes an angle θ with the positive real axis, then θ is called an argument of z .

For $a \neq 0$,

$$\tan \theta = \frac{b}{a}.$$

Thus,

$$\theta = \tan^{-1} \left(\frac{b}{a} \right),$$

but the quadrant must always be considered.

Definition

The principal argument is normally chosen such that

$$-\pi < \theta \leq \pi.$$

The argument is not unique. If θ is an argument, then

$$\theta + 2n\pi, \quad n \in \mathbb{Z},$$

is also an argument.

Solved Example

Find the principal argument of

$$z = -1 + \sqrt{3}i.$$

The point lies in the second quadrant.

We have

$$\tan \theta = -\sqrt{3}.$$

Therefore,

$$\theta = \frac{2\pi}{3}.$$

10 Cartesian Representation

A complex number

$$z = a + ib$$

can be represented by the point

$$P(a, b)$$

in a plane.

The horizontal axis is called the real axis and the vertical axis is called the imaginary axis.

Thus,

$$z = a + ib \longleftrightarrow P(a, b).$$

If $OP = r$, then

$$r = \sqrt{a^2 + b^2} = |z|.$$

If OP makes an angle θ with the positive real axis, then

$$a = r \cos \theta,$$

and

$$b = r \sin \theta.$$

These relations lead directly to the polar form of a complex number.

11 Polar or Trigonometric Representation

Let

$$z = a + ib.$$

Put

$$r = |z|.$$

If θ is an argument of z , then

$$a = r \cos \theta$$

and

$$b = r \sin \theta.$$

Therefore,

$$z = r \cos \theta + ir \sin \theta.$$

Hence,

$$z = r(\cos \theta + i \sin \theta).$$

The expression

$$\cos \theta + i \sin \theta$$

may also be written as

$$\text{cis } \theta.$$

Thus,

$$z = r \text{cis } \theta.$$

Solved Example

Express

$$z = -1 + i$$

in polar form.

First,

$$r = \sqrt{(-1)^2 + 1^2} = \sqrt{2}.$$

The point lies in the second quadrant, so

$$\theta = \frac{3\pi}{4}.$$

Hence,

$$z = \sqrt{2} \left(\cos \frac{3\pi}{4} + i \sin \frac{3\pi}{4} \right).$$

12 Conversion Between Cartesian and Polar Forms

12.1 Cartesian to Polar

Given

$$z = a + ib,$$

calculate

$$r = \sqrt{a^2 + b^2}.$$

Then determine θ by considering the quadrant.

Finally write

$$z = r(\cos \theta + i \sin \theta).$$

12.2 Polar to Cartesian

Given

$$z = r(\cos \theta + i \sin \theta),$$

we have

$$a = r \cos \theta$$

and

$$b = r \sin \theta.$$

Solved Example

Convert

$$z = 2 - 2\sqrt{3}i$$

into polar form.

We have

$$r = \sqrt{2^2 + (-2\sqrt{3})^2} = 4.$$

The point lies in the fourth quadrant, so

$$\theta = -\frac{\pi}{3}.$$

Therefore,

$$z = 4 \left(\cos \left(-\frac{\pi}{3} \right) + i \sin \left(-\frac{\pi}{3} \right) \right).$$

13 De Moivre's Theorem

Definition

For an integer n ,

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta.$$

This is called De Moivre's theorem.

If

$$z = r(\cos \theta + i \sin \theta),$$

then

$$z^n = r^n(\cos n\theta + i \sin n\theta).$$

Thus,

$$\boxed{[r(\cos \theta + i \sin \theta)]^n = r^n(\cos n\theta + i \sin n\theta)}.$$

Solved Example

Evaluate

$$(\cos 30^\circ + i \sin 30^\circ)^4.$$

By De Moivre's theorem,

$$= \cos 120^\circ + i \sin 120^\circ.$$

Therefore,

$$\boxed{-\frac{1}{2} + \frac{\sqrt{3}}{2}i}.$$

14 Applications of De Moivre's Theorem

De Moivre's theorem is useful for calculating powers of complex numbers.

Suppose

$$z = r(\cos \theta + i \sin \theta).$$

Then

$$z^n = r^n(\cos n\theta + i \sin n\theta).$$

It can also be used to obtain multiple-angle formulae.

For $n = 2$,

$$(\cos \theta + i \sin \theta)^2 = \cos 2\theta + i \sin 2\theta.$$

Expanding the left side,

$$\cos^2 \theta - \sin^2 \theta + 2i \sin \theta \cos \theta.$$

Comparing real parts,

$$\boxed{\cos 2\theta = \cos^2 \theta - \sin^2 \theta}.$$

Comparing imaginary parts,

$$\sin 2\theta = 2 \sin \theta \cos \theta.$$

15 Polynomial Fractions

A polynomial fraction is a rational expression of the form

$$\frac{P(x)}{Q(x)},$$

where $P(x)$ and $Q(x)$ are polynomials and

$$Q(x) \neq 0.$$

There are two important types.

15.1 Proper Fraction

If

$$\deg P(x) < \deg Q(x),$$

then the fraction is proper.

Example:

$$\frac{x+1}{x^2+3x+2}.$$

15.2 Improper Fraction

If

$$\deg P(x) \geq \deg Q(x),$$

then the fraction is improper.

Example:

$$\frac{x^2+3x+2}{x+1}.$$

Important Point

An improper fraction must first be simplified using polynomial division before resolving the remaining proper fraction into partial fractions.

16 Partial Fractions

Partial fraction decomposition expresses a rational expression as a sum of simpler rational expressions.

For example,

$$\frac{1}{(x+1)(x+2)}$$

can be written as

$$\frac{A}{x+1} + \frac{B}{x+2}.$$

The constants A and B are determined by comparing coefficients or substituting convenient values of x .

Partial fractions are particularly useful when simplifying rational expressions and in applications such as integration.

17 Partial Fractions with Distinct Linear Factors

Suppose

$$\frac{P(x)}{(x-a)(x-b)}$$

is a proper fraction.

Write

$$\frac{P(x)}{(x-a)(x-b)} = \frac{A}{x-a} + \frac{B}{x-b}.$$

Multiplying by the denominator gives

$$P(x) = A(x-b) + B(x-a).$$

The constants can then be determined.

Solved Example

Resolve

$$\frac{5x + 7}{(x + 1)(x + 2)}$$

into partial fractions.

Write

$$\frac{5x + 7}{(x + 1)(x + 2)} = \frac{A}{x + 1} + \frac{B}{x + 2}.$$

Multiplying throughout,

$$5x + 7 = A(x + 2) + B(x + 1).$$

Putting $x = -1$,

$$2 = A.$$

Putting $x = -2$,

$$-3 = -B,$$

so

$$B = 3.$$

Therefore,

$$\boxed{\frac{5x + 7}{(x + 1)(x + 2)} = \frac{2}{x + 1} + \frac{3}{x + 2}}.$$

18 Partial Fractions with Repeated Factors

Suppose the denominator contains

$$(x - a)^n.$$

Then the partial fraction decomposition contains all powers:

$$\frac{A_1}{x - a} + \frac{A_2}{(x - a)^2} + \cdots + \frac{A_n}{(x - a)^n}.$$

For example,

$$\frac{P(x)}{(x - a)^3} = \frac{A}{x - a} + \frac{B}{(x - a)^2} + \frac{C}{(x - a)^3}.$$

Solved Example

Resolve

$$\frac{1}{x(x+1)^2}.$$

Write

$$\frac{1}{x(x+1)^2} = \frac{A}{x} + \frac{B}{x+1} + \frac{C}{(x+1)^2}.$$

Multiplying by

$$x(x+1)^2,$$

we get

$$1 = A(x+1)^2 + Bx(x+1) + Cx.$$

Putting $x = 0$,

$$A = 1.$$

Putting $x = -1$,

$$C = -1.$$

Comparing coefficients gives

$$B = -1.$$

Therefore,

$$\boxed{\frac{1}{x(x+1)^2} = \frac{1}{x} - \frac{1}{x+1} - \frac{1}{(x+1)^2}}.$$

19 Partial Fractions with Quadratic Factors

If an irreducible quadratic factor occurs in the denominator, its numerator must be linear.

For example,

$$\frac{P(x)}{(x-a)(x^2+px+q)}$$

has the form

$$\frac{A}{x-a} + \frac{Bx+C}{x^2+px+q}.$$

The numerator over the quadratic factor is

$$Bx + C$$

rather than a constant.

Important Point

Remember:

Linear factor $\Rightarrow$ constant numerator

Quadratic factor $\Rightarrow$ linear numerator

20 Resolution of Improper Fractions

For an improper rational expression

$$\frac{P(x)}{Q(x)},$$

first perform polynomial division.

We obtain

$$\frac{P(x)}{Q(x)} = S(x) + \frac{R(x)}{Q(x)},$$

where

$$\deg R(x) < \deg Q(x).$$

The remaining proper fraction can then be resolved into partial fractions.

Solved Example

Consider

$$\frac{x^2 + 3x + 2}{x + 1}.$$

Dividing,

$$x^2 + 3x + 2 = (x + 1)(x + 2).$$

Therefore,

$$\frac{x^2 + 3x + 2}{x + 1} = x + 2.$$

Thus the original improper fraction reduces to

$$x + 2.$$

21 General Method for Partial Fractions

The following procedure is useful for most problems.

1. Check whether the fraction is proper or improper.
2. If improper, perform polynomial division.
3. Factorise the denominator.
4. Identify the types of factors.
5. Write the appropriate partial fraction form.
6. Multiply both sides by the common denominator.
7. Determine the unknown constants.
8. Substitute the constants into the decomposition.
9. Simplify the final answer.
10. Verify the result if necessary.

22 Important Formula Sheet

Complex Numbers

$$z = x + iy$$

$$i^2 = -1$$

$$\bar{z} = x - iy$$

$$|z| = \sqrt{x^2 + y^2}$$

$$z\bar{z} = |z|^2$$

$$z = r(\cos \theta + i \sin \theta)$$

$$r = |z|$$

$$x = r \cos \theta$$

$$y = r \sin \theta$$

$$\arg z = \theta + 2n\pi$$

$$-\pi < \text{Arg}(z) \leq \pi$$

$$(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$$

Partial Fractions

Proper fraction:

$$\deg P < \deg Q$$

Improper fraction:

$$\deg P \geq \deg Q$$

Distinct linear factors:

$$\frac{A}{x-a} + \frac{B}{x-b}$$

Repeated linear factor:

$$\frac{A}{x-a} + \frac{B}{(x-a)^2} + \cdots$$

Irreducible quadratic:

$$\frac{Ax+B}{x^2+px+q}$$

23 Practice Questions

Practice Questions

Complex Numbers

1. Find

$$i^{35}.$$

2. Find the real and imaginary parts of

$$7 - 4i.$$

3. Simplify

$$(2 + 3i) + (5 - 7i).$$

4. Simplify

$$(3 + 2i)(4 - i).$$

5. Simplify

$$\frac{4 + 3i}{2 - i}.$$

6. Find the conjugate of

$$6 - 5i.$$

7. Find the modulus of

$$8 - 6i.$$

8. Find the principal argument of

$$-1 + i.$$

9. Express

$$1 + i$$

in polar form.

10. Express

$$2 \left(\cos \frac{\pi}{3} + i \sin \frac{\pi}{3} \right)$$

in Cartesian form.

11. Evaluate using De Moivre's theorem:

$$(\cos 45^\circ + i \sin 45^\circ)^4.$$

Practice Questions**Partial Fractions**

1. Resolve

$$\frac{3x + 5}{(x + 1)(x + 2)}$$

into partial fractions.

2. Resolve

$$\frac{1}{x(x + 2)}$$

into partial fractions.

3. Resolve

$$\frac{2x + 1}{x(x + 1)^2}$$

into partial fractions.

4. Resolve

$$\frac{5x + 3}{(x - 1)(x + 2)}$$

into partial fractions.

5. Identify whether

$$\frac{x + 1}{x^2 + 3x + 2}$$

is proper or improper.

6. Explain the steps required to resolve an improper fraction.

24 Final Revision

Before solving problems from this unit, remember the following points.

1. Always remember

$$i^2 = -1.$$

2. Write every complex number in the form

$$x + iy.$$

3. Identify the real and imaginary parts separately.

4. Use the conjugate while dividing complex numbers.

5. Use

$$|z| = \sqrt{x^2 + y^2}$$

to calculate the modulus.

6. While finding an argument, always check the quadrant.

7. The polar form is

$$z = r(\cos \theta + i \sin \theta).$$

8. Use De Moivre's theorem for powers of complex numbers.

9. For partial fractions, factorise the denominator first.

10. If the fraction is improper, perform polynomial division first.

11. Repeated factors require all corresponding powers.

12. A quadratic factor requires a linear numerator.

Important Point

Complex Numbers: Algebraic Form $z = x + iy$

Polar Form $z = r(\cos \theta + i \sin \theta)$

De Moivre $(\cos \theta + i \sin \theta)^n = \cos n\theta + i \sin n\theta$

Partial Fractions Factorise first, then decompose